Journal of Business and Economics, ISSN 2155-7950, USA
January 2015, Volume 6, No. 1, pp. 1-12
DOI: 10.15341/jbe(2155-7950)/01.06.2015/001

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The Assignment Problem: Searching for An Optimal and Efficient Solution

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Abstract: The 2-dimensional assignment problem, which consists of assigning n objects to n (or m) opportunities in an optimal way, has long been viewed as a special case of the Linear Programming problem. But solving the Assignment problem as a Linear Programming problem is to use, perhaps, the most inefficient method possible. Many other methods are available for solving the 2-dimensional assignment problem more efficiently. This paper briefly discusses several of these methods and then ranks them according to their efficiency, where efficiency is measured by the number of operations needed by each method to complete the assignment.

Key words: assignments; available methods and comparisons; cost matrix; comparative study; complexity and sparcity; optimal and efficient solutions

JEL codes: C02, C19, D79

1. Introduction

The 2-dimensional algorithms discussed in this paper can be used to solve the assignment problem on their own, or they can be viewed as the last step of a more general "K-dimensional assignment problem" methodology in which the information of K sets of data is used to solve the assignment problem instead of the usual one (1) set of data. For example, the solution to the data association problem of Multitarget Tracking can be obtained by:

(1) Processing the information of one data set (i.e., 1 scan) which, even though is real-time, often leads to incorrect assignments of reported information and established airplane tracks (2-dimensional problem), and:

(2) Processing the information of K data sets (i.e., K scans, where $K \geq 1$) which achieves much higher percentages of correct assignments in real time (K-dimensional problem).

The dimensionality of the K-dimensional problem is determined by the number of data sets (K) used. This problem is solved recursively, by reducing the K-dimensional problem to a (K-1)-dimensional problem, by incorporating one set of constraints into the Objective Function, using a set of Lagrangian Multipliers. Then, given a solution of the K-1 dimensional problem, a Feasible Solution of the K-dimensional problem is reconstructed. The K-1 dimensional problem is solved in a similar manner, and the process is repeated until it reaches the 2-dimensional problem which can be solved optimally, or nearly optimally, using one of the several 2-dimensional algorithms discussed in this paper.

The reason there are so many 2-dimensional algorithms is the fact that these algorithms have different degrees of accuracy and computational complexity and, as a consequence, their potential applications should be carefully examined.

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2. Discussion

2.1 Two-Dimensional (Single Scan) Data Assignment Algorithms

The 2-dimensional assignment problem occurs when n facilities are assigned to m objects (jobs) on a one-to-one basis. When n = m, the assignment problem is called SYMMETRIC, while when n ≠ m the assignment problem is called ASYMMETRIC. The assignment is made with the objective of minimizing the overall cost of completing the jobs, or, alternatively, of maximizing the overall profit from the jobs.

A typical illustration, even though very small, of an assignment problem is given by the example below. This problem can be solved by each of the six (6) 2-dimensional algorithms discussed in this paper, but only the solution using the COMPLETE ENUMERATION method is explicitly given. The solution, using the other methods can be obtained by using the attached flow charts for each method. The starting point for each of these 2-dimensional assignment algorithms is a COST MATRIX, a specific example of which is given by the cost matrix shown below.

EXAMPLE: A firm has 3 jobs that need to be assigned to 3 work crews. Because of varying experience of the work crews, each work crew is not able to complete each job with the same effectiveness.

The cost of each work crew to do each job is given by the cost matrix shown below in Table 1.

Table 1 Cost Matrix

		Job (j)		
CREW (i)		1	2	3
	1	41	72	39
	2	22	29	49
	3	27	39	60

The objective is to assign the jobs to the work crews so as to minimize the total cost of completing all jobs.

The six (6) 2-dimensional algorithms discussed in this paper are:

(1) The COMPLETE ENUMERATION method

(2) The SIMPLEX LINEAR PROGRAMMING method

(3) The ASSIGNMENT method

(4) The HUNGARIAN method

(5) The MUNKRES method

(6) The DEEPEST HOLE method

(1) The most direct way of solving a 2-dimensional assignment problem would be a “complete enumeration” of all possible assignments of facilities to objects, the calculation of cost of each assignment, and the identification of the OPTIMUM assignment, which is the assignment with the MINIMUM cost.

The number N of possible assignments of n facilities to m objects (jobs) on a one-to-one basis is equal to:

$$N = \frac{n!}{(n-m)!}, \text{ if } n > m \quad (1)$$

$$\text{and } N = n! \quad \text{if } n = m \quad (2)$$

Therefore when n = m = 3, this number is equal to 3! = 6; when n = m = 5, this number is equal to 5! = 120, while when n = m = 10, the number of possible assignments is equal to 10! = 3,628,000. The Flow Chart for this method, showing the specific steps needed to apply this method and the results of the solution of this example by this method, is shown in Figure 1, of Appendix I.

Quite obviously, enumerating all possible assignments is feasible only for very small problems and, therefore, it is necessary to investigate alternative solution techniques.

(2) Since the assignment problem can be considered a special case of the Linear Programming problem, it can be solved as a Linear Programming problem using the SIMPLEX Algorithm. If c_{ij} is defined as the cost of assigning facility i to job j and x_{ij} is defined as the proportion of time that facility i is assigned to job j, the linear programming problem is:

$$\text{Minimize: } z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \tag{3}$$

$$\text{Subject to: } \sum_{i=1}^{n} x_{ij} = 1 \quad \text{for } j = 1, 2, \ldots, n \tag{4}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \quad \text{for } i = 1, 2, \ldots, n \tag{5}$$

$$x_{ij} \geq 0 \tag{6}$$

But this approach involves much computational burden. In fact, it took 5 Linear Programming Tableaus to successfully solve the problem stated above which is a rather simple problem. In general, the number of Tableaus needed to solve a Linear Programming problem cannot exceed T_{max}, where:

$$T_{max} = \frac{n!}{m!(n-m)!} \tag{7}$$

m = The number of constraining equations in the Linear Programming Problem,

And n = The number of variables in the Linear Programming Formulation of the problem

The Flow Chart for the SIMPLEX method is given in Figure 2, of Appendix I.

It is not clear how this maximum number of Tableaus translates into the "maximum number of operations" needed to solve the assignment problem so that the computational efficiency of this method could be compared to the computational efficiency of other methods, such as the MUNKRES Algorithm, where the computation efficiency is related to the size of the Square Cost matrix.

What is needed for a meaningful comparison is a complete Comparative study in which the "Computational Efficiency" and "Time-to-Solution" of each of these methods can be obtained as a function of the dimensionality (size) of the Cost matrix.

The: ASSIGNMENT, HUNGARIAN, and MUNKRES Algorithms, are related (to some extent) to each other because each of them, starting with the given Cost matrix, attempts to find the OPTIMAL assignment by inducing the "relative costs" of the Facilities/Jobs pairings to zero, through appropriate "manipulation" of the Rows and/or Columns of the given Cost matrix. Their differences are mainly in the way this "manipulation" is carried out, but the ASSIGNMENT and MUNKRES Algorithms can also be used to solve non-symmetric problems while the HUNGARIAN Algorithm cannot.

(3) The ASSIGNMENT Algorithm is based on two facts, namely:

(a) Each Facility must be assigned to one of the jobs.

(b) The Relative Cost of assigning Facility *i* to Job *j* is not changed by the subtraction of a constant from either a column or a row of the Cost matrix.

It arrives at an OPTIMAL assignment when the Total Relative Cost of the assignment is zero. This method can also be used to solve MAXIMIZATION Assignment problems by first converting the given cost matrix into a "REVERSED MAGNITUDES" Cost matrix, and NON-SYMMETRIC problems. The Flow Chart of the ASSIGNMENT Algorithm is shown in Figure 3, of Appendix I.

(4) The HUNGARIAN method (or Kuhn's Algorithm) is related to both, the previously discussed ASSIGNMENT Algorithm, and the MUNKRES Algorithm, which follows. The method consists of four (4) basic steps, and it is an iterative procedure because some of the steps have to be repeated. The method uses a "minimal set of lines" to cover the zeros of the "manipulated" Cost matrix, and the procedure terminates when the required "minimal set of lines" is equal to the dimensionality of the given Cost matrix. The Flow Chart of the HUNGARIAN method is given in Figure 4, of Appendix I.

(5) The MUNKRES Algorithm is an OPTIMAL Assignment Algorithm, which can be considered a variant of the HUNGARIAN Algorithm. It differs from it in the detailed procedures for finding:

(a) The "minimal set of lines" which contain all the zeros.

(b) The "maximal set" of independent zeros.

An important feature of this algorithm is the fact that its inventor, in his paper introducing the algorithm, also derived an equation to calculate the "maximum number of operations needed to solve completely any nxn assignment problem". This equation is given by:

$$N_{max} = (n/6)\ (11n^2 + 12n + 31) \tag{8}$$

This maximum is of both theoretical and practical interest because it is much smaller (for $n \geq 6$) than the n! operations necessary to solve the assignment problem using the COMPLETE ENUMERATION method. But N_{max} is larger than the number of operations needed to solve the assignment problem by some of the other methods discussed in this paper (the DEEPEST HOLE Algorithm, for example). There is, however, a problem in such comparisons because of the difficulty encountered in defining a "standard" operation in each of the algorithms. What is needed to obtain an accurate comparison of the Computational Complexity of each of the algorithms discussed in this paper is a complete comparative study in which the "time to obtain the solution to the problem" and the "accuracy of assignment" are evaluated as a function of the same Input Cost matrix, where the "size" of the matrix and the "sparcity" of the matrix are allowed to vary, but the same "size and sparcity" Cost matrix serves as the input to all six (6) 2-dimensional algorithms discussed in this paper. The Flow Chart of the MUNKRES Algorithm is shown in Figure 5, of Appendix I.

(6) The DEEPEST HOLE Algorithm, is a "relatively" simple Algorithm and it is computationally more efficient than the OPTIMAL MUNKRES Algorithm. However, there is some loss in performance because it cannot be guaranteed to always select the lowest overall cost assignment. If this "loss in association" can be kept to a minimum, the benefits derived from this algorithm (because of its computational speed and simplicity of implementation) may out-weigh its non-optimality. The Algorithm consists of four (4) simple basic steps, and its iterative because the procedure continues until all pairs have been assigned. The Flow Chart of the DEEPEST HOLE Algorithm is shown in Figure 6, of Appendix I.

2.2 Comparing the Two-dimensional Assignment Algorithms

As already stated in the Introduction section of this paper, the K-dimensional (i.e., K-scan) methodology of data assignment utilizes a 2-dimensional (i.e., single-scan) assignment in its "terminal phase". IF THE OBJECTIVE IS TO DESIGN THE "MOST EFFICIENT" K-dimensional Algorithm, we must select the "best" 2-dimensional algorithm available to support the K-dimensional algorithm. But even if we opted to use a single-scan algorithm to solve the assignment problem, again we need to select the "best" 2-dimensional algorithm available. But, what is meant by “best”, and which of the 2-dimensional algorithms is the “best”? To answer the first part of this question it is necessary to first establish our criteria of “goodness”. Accuracy (of assignment) and Computational Complexity appear to be two such criteria, but they are not independent. In general, Computational Complexity increases with Accuracy requirements, with “perfect” accuracy (i.e. OPTIMAL solution) resulting in more computational complexity, but the exact relationship between these two criteria is not simple, and it is not easy to obtain.

If Accuracy of assignment is the only criterion used, then both the “COMPLETE ENUMERATION” method and the MUNKRES Algorithm are “equally good” because they both result in the OPTIMAL solution. However, when Computational Complexity is also considered as a criterion, the MUNKRES Algorithm is “better” because it requires fewer operations to arrive at the optimal solution than the “COMPLETE ENUMERATION” method, for $n \geq 6$. This conclusion is drawn from the fact that, if n is the size of the square Cost matrix, the “COMPLETE ENUMERATION” method requires n! “operations” to arrive at the optimal solution while the maximum number of “operations” required by the MUNKRES Algorithm to arrive at the optimal solution is:

$$N_{max} = (n/6)\ (11n^2 + 12n + 31) \tag{9}$$

and

$$n! \geq N_{max} \qquad \text{for: } n \geq 6 \tag{10}$$

However, a potential problem exists even with this apparently flawless logic! What is an “operation”, and are the “operations” of the two (2) algorithms the same? From the six (6) 2-dimensional Assignment Algorithms discussed in this paper, it appears that the DEEPEST HOLE Algorithm is the SIMPLEST (i.e., has the least computational complexity). But, as we have already stated, this Algorithm does not always produce the OPTIMAL solution! How much accuracy are we willing to sacrifice to gain a measure of computational SIMPLICITY?

The other 2-dimensional Assignment Algorithms, appear to fall somewhere between the MUNKRES and the DEEPEST HOLE Algorithms when Computational Complexity is considered as the primary criterion of “goodness”, since the other Algorithms (ASSIGNMENT, HUNGARIAN) are variations of the MUNKRES Algorithm.

How would each of the six (6) 2-dimensional Assignment Algorithms affect the Accuracy and Computational Complexity of the K-dimensional algorithm? We really do not know! Aubrey Poore of CSU (Colorado State University) claims that, for an nxn square Cost matrix, the Computational Complexity of the K-dimensional algorithm, is less than n^3. But, unfortunately, he has not evaluated the Computational Complexity of the K-dimensional Algorithm with all of the 2-dimensional Algorithms discussed in this paper, in its “terminal phase”.

The Computational Complexity of some algorithms is shown in Table 2 and summarized below:

(1) K-Dimensional “LaGrangian Relaxation” Method $< n^3$

(2) 2-Dimensional “Munkres” Method $\sim n^3$

(3) 2-Dimensional “Pure Combinatorial” Method $\sim n!$

Obviously, what is needed to assess the relative merits of each of the six (6) 2-dimensional Assignment Algorithms, and the effect that each one of them has on the Accuracy and Computational Complexity of the K-dimensional algorithm, is a complete comparative study in which these two criteria of “goodness” (i.e., accuracy of assignment, and computational complexity) for each of the six 2-dimensional Algorithms will be evaluated as a function of the DIMENSIONALITY of the Cost matrix (i.e., nxn or nxm) and the SPARCITY (i.e., the number of zero elements) of the Cost matrix.

Table 2 Computational Complexity of Algorithms

n	n^2	n^3	n!
1	1	1	1
2	4	9	2
3	9	27	6
4	16	64	24
5	25	125	120
6	36	216	720
7	49	343	5,040
8	64	512	40,320
9	81	729	362,880
10	100	1,000	3,628,800

3. Conclusions and Recommendations

(1) The Assignment Problem, considered a special case of the Linear Programming Problem (LPP) can be either "K-Dimensional" or "2-Dimensional".

(2) There are many "2-Dimensional" methods available, and six of them are discussed in this paper.

(3) Since the objective is to select the "Best" Assignment Algorithm, whether it is "K-Dimensional" or "2-Dimensional", we need to define "Criteria of Goodness" to help us in this evaluation.

(4) The criteria of goodness selected are:

(a) Accuracy of Assignment, and

(b) Computational Complexity of the method

(5) The assignment methods discussed have varying degrees of accuracy and computational complexity.

(6) Selecting the "Best" "2-Dimensional" method, which can be used by itself, or as the terminal phase of a "K-Dimensional" algorithm, is not easy, because the "Computational Complexity" of "2-Dimensional Methods" depend on the size (n) of the matrix.

(7) What is needed, to assess the relative merits of each "2-Dimensional" method, and the effect each has on the Accuracy and "Computational Complexity" of the "K-Dimensional" algorithm, is a complete comparative study in which the 2 "criteria of goodness" will be evaluated as a function of the "Dimensionality" of the cost matrix (n x n or m x n) and the "Sparcity" of the cost matrix (i.e., the number of zero elements in the cost matrix).

(8) At the conclusion of such a complete comparative study, we will be able to identify the "Best" "2-Dimensional" method, and also identify the "Best" "K-Dimensional" algorithm by attaching the "Best" "2-Dimensional" method to the terminal phase of the "K-Dimensional" algorithm.

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Appendix I Flow Charts of the 2-Dimensional Assignment Methods

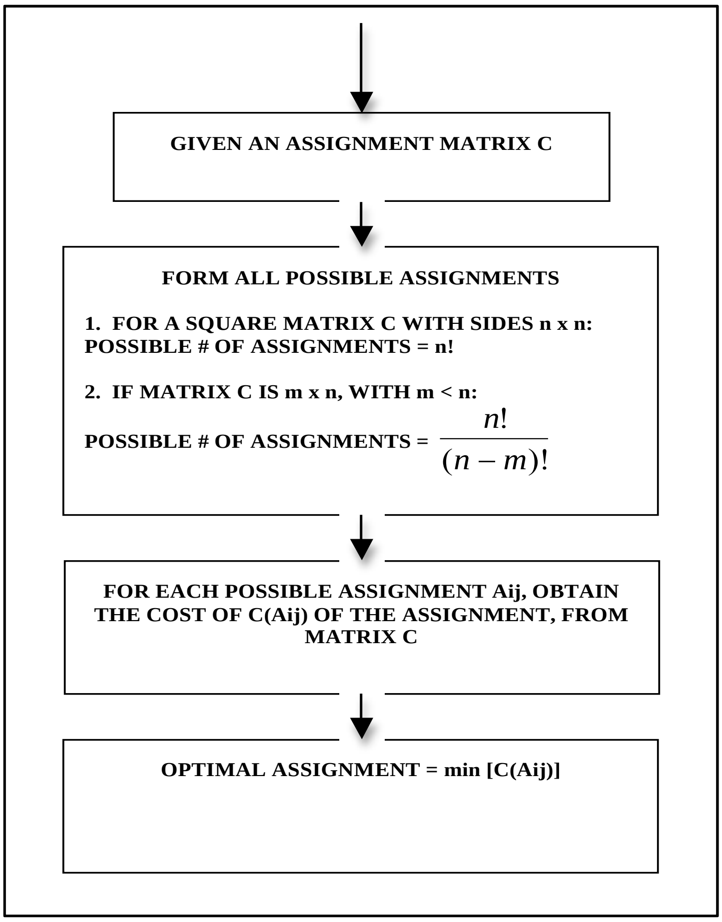

Figure 1 Complete Enumeration Method

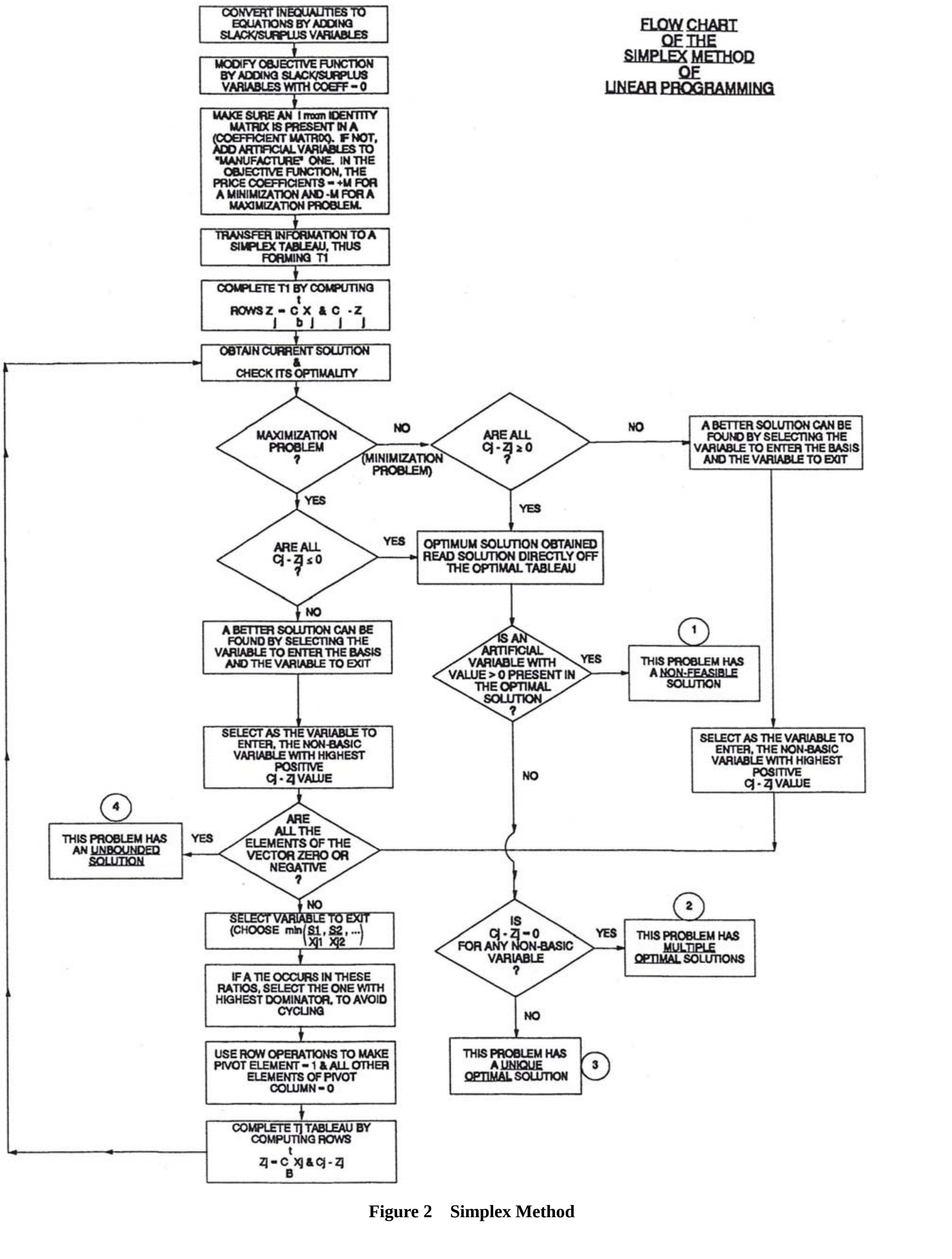

Figure 2 Simplex Method

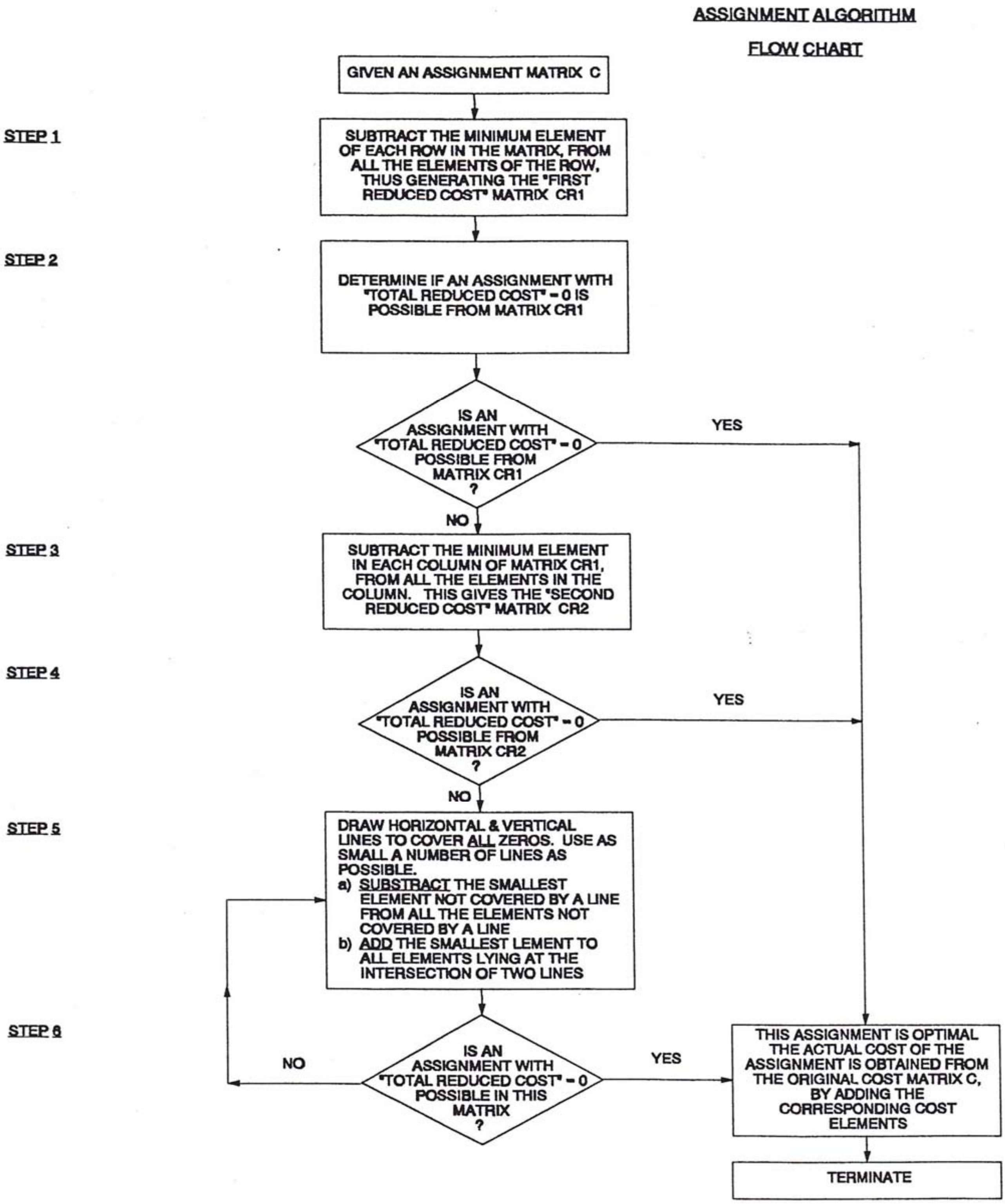

Figure 3 Assignment Method

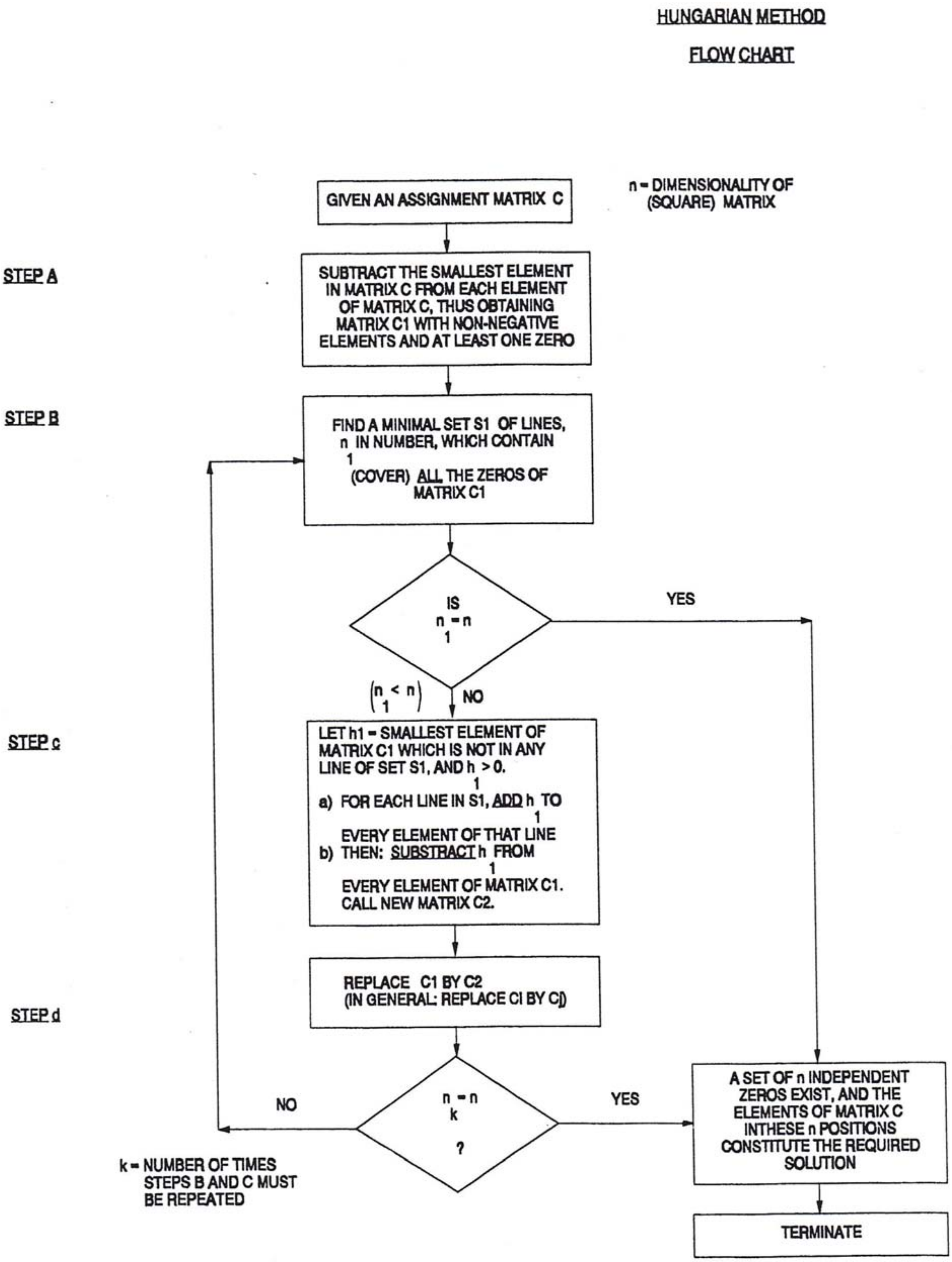

Figure 4 Hungarian Method

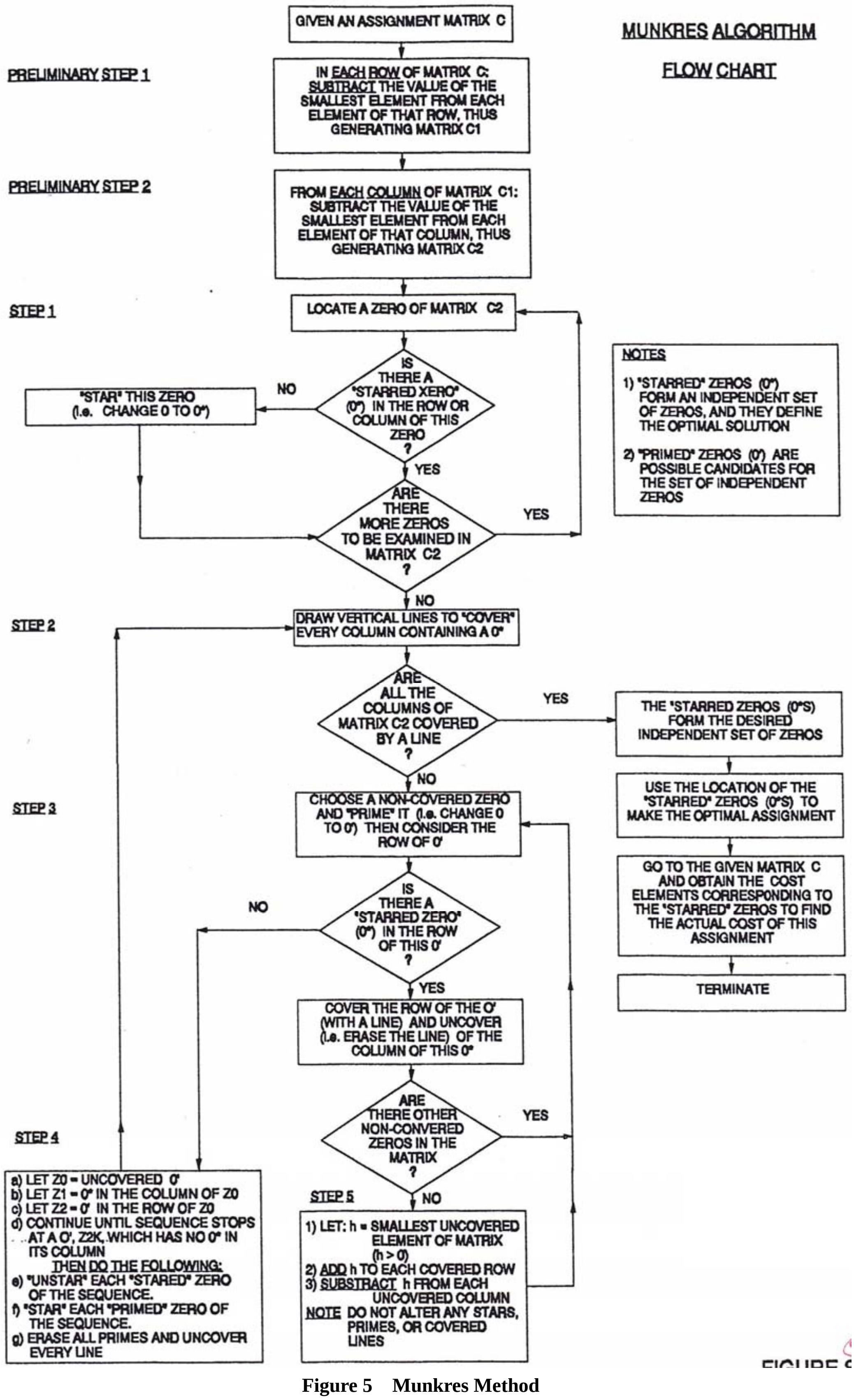

Figure 5 Munkres Method

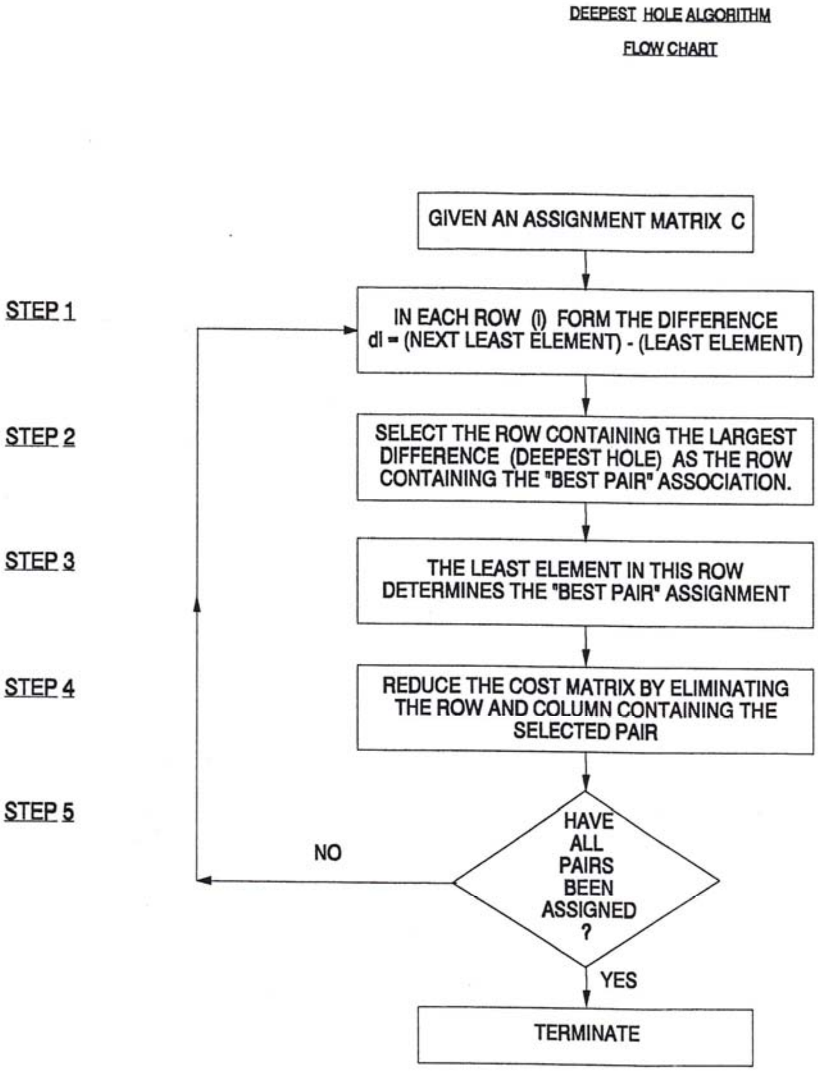

Figure 6 Deepest Hole Method