Modern Environmental Science and Engineering (ISSN 2333-2581)
Octoer 2020, Volume 6, No. 10, pp. 1078-1084
Doi: 10.15341/mese(2333-2581)/10.06.2020/006
Academic Star Publishing Company, 2020
www.academicstar.us

Estimation of Settlements in Shallow Foundations Based on the Theory of Linear Viscoelasticity

Juliane Cristina Gonçalves

Federal University of Juiz de Fora, Brazil

Abstract: In this work, the main focus is a methodology for the analysis of the settlements of a spread footing underneath the center of compression load area applying the Theory of Linear Viscoelasticity. A review of the foundations of Linear Viscoelasticity and the development leading to the equation for solving the problem are briefly presented. It was possible to represent the viscoelasticity behavior of materials through rheological models formed by a combination of springs, dashpots and friction blocks. Their creep functions may be adopted to model in approximate value the behavior of materials or structural elements and foundations. The rheological model in this study consists of a spring in series with a Kelvin model (a spring and a dashpot in parallel). The method is illustrated with an example as following: footing of width 1.2 m and average applied stress 174.2 kN/m². Settlement growth is observed over time. The estimated values are always very small, reaching only 6.4 mm. It is already expected such a low value due to footing width, average applied stress, coefficient of viscosity, transverse elastic modulus, Poisson's ratio and soil elastic modulus. Settlements tend toward stabilizing after about 300 days.

Key words: estimation of settlements, shallow foundation, footing foundation, linear viscoelasticity

1. Introduction

Rheology is the science dealing with the deformation and flow of matter. Deformation refers to the change in shape and dimensions of the amount of matter, under the influence of external forces. The flow of matter is associated with the portion of the deformation which is time dependent.

A didactical manner of representing the viscoelastic behavior of materials is to apply rheological models constituted by a combination of springs, dashpots and friction blocks. Their creep functions may be employed to approximate the behavior of materials or structural elements and foundations.

When a material exhibits linear viscoelastic behavior, in an any given time, the stress is proportional to rate of strain and the Boltzmann superposition principle is valid, represented by the following expressions:

$$\varepsilon\left[C\sigma(t)\right]=C\varepsilon\left[\sigma(t)\right] \tag{1}$$

$$\varepsilon\left[\sigma_1(t)+\sigma_2(t-t_1)\right]=\varepsilon\left[\sigma_1(t)\right]+\varepsilon\left[\sigma_2(t-t_1)\right] \tag{2}$$

where:

σ = applied stress;

ε = resulting deformation;

C = constant.

Creep functions exert the same function in Viscoelasticity comparing to elastic parameters in Elasticity. It means that they establish the relation between stress and deformations including the time variable in this context. Thus, a viscoelastic problem is defined if all creep functions are known, besides the boundary conditions. Creep tests, empirical expressions or mathematical modeling of phenomena may be used to determine these functions.

This work presents the estimation of settlements of a spread footing underneath the center of a compression

Corresponding author: Juliane Cristina Gonçalves, D.Sc.; research areas/interests: structures and foundations. E-mail: juliane.goncalves@ufjf.edu.br.

load area applying the Theory of Linear Viscoelasticity. Therefore we reach an outcome over time.

2. Fundamentals of Linear Viscoelasticity

Different authors provide definitions about the fundamentals of Linear Viscoelasticity [1-3]. The following is a brief review of the fundamentals of Linear Viscoelasticity [4-6].

2.1 Concept of Creep and Relaxation

Creep is a type of slow and permanent deformation occuring in certain materials due to exposure to constant stress and temperature. Creep can be subdivided into: (i) primary, exhibiting a decreasing strain rate, (ii) secondary, when the deformation speed is constant, (iii) tertiary, when the creep is accelerating until fracture.

Relaxation is the gradual reduction in stress when a material is loaded and then held at a constant level of strain, under constant temperature.

2.2 Viscoelastic Models

To acquire a better understanding of the macroscopic mechanisms governing the behavior of a real system, it is a common practice to replace that system with an ideal mechanical model called a rheological model. The rheological models consist of elementary units of springs, dashpots and friction blocks, connected in series or in parallel.

The elementary models are:

(i) Hookean (consisting of a single spring, with linear behavior, and it is assumed that stress response is independent of time);

(ii) Newtonian (consisting of a single dashpot, with linear behavior, with a time-dependent response);

(iii) Rigid-Plastic (consisting of a single friction block, with stress-strain responses proportional to the step function, with a time-independent response).

Considering that stress versus strain versus time relationships of many materials do not usually follow the stress versus strain versus time pattern of elementary models, a combination of these models is required to represent the behavior of these materials. These models called elementary and composite rheological models are:

(i) Saint-Venant model (composed by a combination of a spring and a friction block in series);

(ii) Kelvin model (composed by a combination of a spring and a dashpot in a parallel combination with a viscoelastic response);

(iii) Maxwell model (composed by a combination of a spring and a dashpot with a viscoelastic response).

Concerning those materials that exhibit more complex behavior, the combination of elementary models and elementary composite models produces a new class of rheological models, called complex composite models.

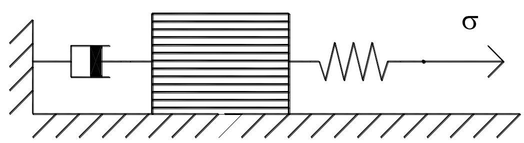
The most typical and widely used are:

(i) Bingham model (the association in series of a dashpot, a spring and a friction block);

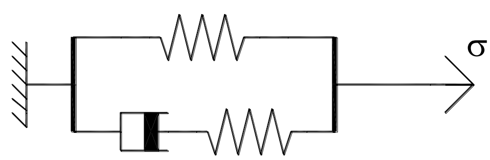
(ii) Standard Linear model (the association of Maxwell model with a linear spring connected in parallel);

(iii) Burgers model (the serial assembly of Maxwell and Kelvin models).

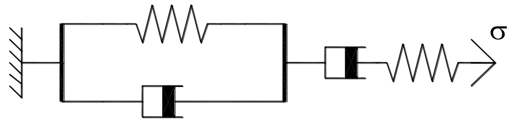
Fig. 1 shows the complex compound rheological models.



(a) Bingham model



(b) Standard Linear Model



(c) Burgers model

Fig. 1 Complex composite rheological models.

2.3 Mathematical Representation of Creep

Various relationships between stress and strain and also time are basically empirical in the technical literature. Most of them were established in a way to adjust to experimental results achieved under constant tension and temperature. However, the real material behavior has revealed that the deformation with time depends on the states of stresss to which material has been subjected in the past, and not on its maximum value. Therefore the creep phenomenon is affected by prior strain history. In light of this fact, several mathematical methods have been suggested to represent the viscoelastic behavior of materials, including the differential and integral forms. The integral form is presented in this work. A great advantage in using the integral representation over the differential one is the flexibility of representation of viscoelastic properties of the material derived directly from tests. This representation may also be used to describe the behavior of materials at aging parameters and incorporate the impacts of temperature. Furthermore concerning those problems in which temporal function of loading is very complex, the integral equation provides a simpler solution than the differential equation [7].

2.3.1 Creep Function

It is accepted that, in a creep test, a step of constant stress $\sigma = \sigma_0 H(t)$ (where H is the Heaviside step function) is applied and the strain $\varepsilon(t)$ is measured. Considering materials with linear behavior, deformations can be represented as follows:

$$\varepsilon(t) = \sigma_0 J(t) \tag{3}$$

Or

$$J(t) = \frac{\varepsilon(t)}{\sigma_0} \tag{4}$$

The function $J(t)$ is called the creep or coupling function and it is a property of materials.

2.3.2 Relaxation Function

Considering that during a relaxation test performed on a material with linear behavior, a deformation $\varepsilon = \varepsilon_0 H(t)$ is prescribed and the stress $\sigma(t)$ is measured, it can be expressed as follows:

$$\sigma(t) = \varepsilon_0 R(t) \tag{5}$$

Or

$$R(t) = \frac{\sigma(t)}{\varepsilon_0} \tag{6}$$

The function $R(t)$ is called the relaxation function. $R(t)$ is a property of materials just like *J(t)*.

2.3.3 Integral Representation of Creep for Uniaxial Stress

If a viscoelastic body with linear behavior is subjected to a constant stress function (t) and a finite derivative on the time interval of interest, representing the stress history, the corresponding deformation function can be obtained from the expression below:

$$\varepsilon(t) = \int_0^t J(t-\tau)\frac{\partial \sigma(\tau)}{\partial \tau} d\tau \tag{7}$$

Where:

τ = auxiliary variable;

$J(t-\tau)$ = creep function.

The expression (7) can also be represented in this manner:

$$\varepsilon(t) = \sigma(t)J(0) - \int_0^t \frac{\partial J(t-\tau)}{\partial \tau}\sigma(\tau)d\tau \tag{8}$$

accepting that $J(t-\tau)$ is continuous in the interval $(0,t)$ and can be derived from τ.

The expressions (7) and (8) are both valid considering the process starts at time $t = 0$ and the initial value of the voltage is null, that is, $\sigma(0) = 0$. Concerning the general case, with the initial value not equal to zero, the following expressions shall apply:

$$\varepsilon(t) = \sigma(\tau_0)J(t-\tau_0) + \int_{\tau_0}^t J(t-\tau)\frac{\partial \sigma(\tau)}{\partial \tau} d\tau \tag{9}$$

$$\varepsilon(t) = \sigma(t)J(0) - \int_{\tau_0}^t \frac{\partial J(t-\tau)}{\partial \tau}\sigma(\tau)d\tau \tag{10}$$

2.4 Relations Between Creep and Relaxation Functions

After studying the creep function, the expressions (7), (8), (9) or (10) could be used to predict stresses based on prescribed deformation history. However, determining $\sigma(t)$ using one of the expressions mentioned above involves solving an integral equation which is expected to be mathematically much more complicated than direct integration. Thus we may write:

$$\sigma(t) = \varepsilon(\tau_0)R(t-\tau_0) + \int_{\tau_0}^{t} R(t-\tau)\frac{\partial \varepsilon(\tau)}{\partial \tau}d\tau$$

(11)

$$\sigma(t) = \varepsilon(t)R(0) - \int_{\tau_0}^{t} \frac{\partial R(t-\tau)}{\partial \tau}\varepsilon(\tau)d\tau \qquad (12)$$

Therefore aiming to determine $\sigma(t)$ based on history of prescribed deformations, it is crucial to know the relaxation function $R(t)$, obtained from a stress relaxation test under a construous strain.

Once the creep and stress relaxation represent two aspects of the same viscoelastic behavior of materials, the creep and relaxation functions are therefore related, that is, one can predict relaxation behavior from known creep parameters and vice-versa. Thus, according to expressions (9) and (10), for $\sigma(0) = 0$ and $\varepsilon(0) = 0$, respectively, we may find for $\tau_0 = 0$:

$$\varepsilon(t) = \int_0^t J(t-\tau)\frac{\partial \sigma(\tau)}{\partial \tau}d\tau \qquad (13)$$

$$\sigma(t) = \int_0^t R(t-\tau)\frac{\partial \varepsilon(\tau)}{\partial \tau}d\tau \qquad (14)$$

Using Laplace transform in the previous equations and eliminating stress-strain transformations and performing then the inverse transform, we can get:

$$\int_0^t J(t-\tau)R(\tau)d\tau = \int_0^t R(t-\tau)J(\tau)d\tau = t \qquad (15)$$

The expression (15) represents the relations between the creep and relaxation functions.

2.5 The Elastic-Viscoelastic Correspondence Principle

If a solution to an elasticity problem is known, the Laplace transform of the solution corresponding to the viscoelastic problem can be obtained by replacing the elastic constants E and $\frac{\nu}{E}$ by $\frac{1}{s\hat{J}_E}(s)$ and $\frac{1}{s\hat{J}_\nu}(s)$, respectively. If the problem is described in terms G and K these constants should be replaced by $\frac{1}{s\hat{J}_G}(s)$ e $\frac{1}{s\hat{J}_K}(s)$.

Creep functions $J_E(t)$, $J_G(t)$ and $J_K(t)$ can be defined as:

Simple traction or compression - axial deformation:

$$\varepsilon_i(t) = \frac{\sigma_i(t)}{E} - \int_0^t \frac{\partial J_E(t-\tau)}{\partial \tau}\sigma_i(\tau)d\tau \qquad (16)$$

Simple traction or compression-transverse deformation:

$$\varepsilon_j(t) = \varepsilon_K(t) = -\frac{\nu}{E}\sigma_i(t) + \int_0^t \frac{\partial J_\nu(t-\tau)}{\partial \tau}\sigma_i(\tau)d\tau \qquad (17)$$

Pure shear-shear deformation:

$$\varepsilon_{ij} = \frac{\sigma_{ij}(t)}{2G} - \int_0^t \frac{1}{2}\frac{\partial J_G(t-\tau)}{\partial \tau}\sigma_{ij}(\tau)d\tau \qquad (18)$$

Hydrostatic compression-volumetric deformation:

$$\varepsilon_{KK}(t) = \frac{\sigma_{KK}(t)}{3K} - \int_0^t \frac{1}{3}\frac{\partial J_K(t-\tau)}{\partial \tau}\sigma_{kk}(\tau)d\tau \qquad (19)$$

Naturally, the viscoelastic solution is thereafter obtained by an inverse Laplace transform.

2.6 The Elastic-Viscoelastic Correspondence Theorems

According to the first Theorem of Correspondence, the internal forces (tensions or internal forces at the sections) caused by imposed load are not modified by creep. At any time *t* the internal forces act upon a body with the same geometric and binding features, with the same loads, but possessing elasticity. Conversely, deformations and displacements showed increases with time following the laws of creep.

The second Theorem of Correspondence shows that the deformations and displacements due to imposed deformations are not modified by creep. However the internal efforts and hyperstatic reactions originting from these imposed deformations decrease over time, according to the laws of relaxation.

3. Example of Application

A methodology is developed to estimate the settlement of a spread footing underneath the center of compression load area applying the Theory of Linear Viscoelasticity (TLV). To achieve this, the Elastic-Viscoelastic Correspondence Principle (EVCP) is applied as follows: if a solution to an elasticity problem is known, the Laplace transform of the solution corresponding to the viscoelastic problem can be obtained by replacing the elastic constants E and $\frac{\nu}{E}$ by $\frac{1}{s}\hat{J}_E(s)$ and $\frac{1}{s}\hat{J}_\nu(s)$, respectively, where the creep functions and their respective Laplace transforms are defined by the expressions:

$$J_E(t) = \frac{1}{E_0} + C(t);\ \widehat{J_E}(s) = \frac{1}{E_0 s} + \hat{C}(s) \quad (20)$$

$$J_V(t) = \frac{\upsilon_0}{E_0} + \mu(t);\ \widehat{J_E}(s) = \frac{\upsilon_0}{E_0 s} + \widehat{\mu}\,(s) \quad (21)$$

We noticed that the solution to the viscoelastic problem involves the inverse Laplace transform. Yet this Principle is only valid if the boundary between the surface on which the external load is applied and the surface where prescribed displacements occur is held constant over time, although loads and displacements may vary.

The steps leading to the equation for solving the problem are then presented.

The rheological model used in this study is represented by the association in series of the Hookean model and the Kelvin model, as shown in Fig. 2.

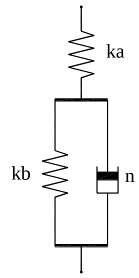


Fig.2 Rheological model adopted.

We highlight some solutions of the Elasticity Theory that allow the estimation of spread footing for a number of cases. For example, the settlements of a spread footing underneath the center of compression load area can be predicted by:

$$r = qBI\left(\frac{1-\nu^2}{E}\right) \quad (22)$$

Where:

q = average applied stress;

B = smaller footing width;

V = Poisson's ratio;

E = elastic modulus of soil;

$$I = I_s I_d I_h$$

I_s= shape factor of the footing and the stiffness;

I_d = depth factor;

I_h = thickness factor of the compressible layer.

From the EVCP, we remark: $\frac{1}{E}$ from the instant solution $\rightarrow \frac{1}{E_0} + s\hat{C}(s)$ in the linear viscoelastic solution and $\frac{\nu}{E} \rightarrow \frac{\nu_0}{E_0} + s\hat{\mu}(s)$, respectively. Thus, $\nu = \frac{\left[\frac{\nu}{E}\right]}{\left[\frac{1}{E}\right]} \rightarrow \frac{\left[\frac{\nu_0}{E_0} + s\hat{\mu}(s)\right]}{\frac{1}{E_0} + s\hat{C}(s)}$. Where the symbol $\rightarrow$ represents corresponds to.

$$\omega(t) = 2C(t) + 2\mu(t) \quad (23)$$

$$\widehat{\omega}(s) = 2\hat{C}(s) + 2\hat{\mu}(s) \quad (24)$$

Hooke's law of elasticity is used for volumetric deformations with:

$$J_K(t) = \frac{1}{K_0} \tag{25}$$

From TLV:

$$J_k(t) = 3\left[J_E(t) - 2J_\nu(t)\right] \tag{26}$$

Developing (26) and applying the respective Laplace transform, this result and the equation (24) generate the following system:

$$\hat{C}(s) - 2\hat{\mu}(s) = \hat{\omega}(s) \tag{27}$$

$$2\hat{C}(s) + 2\hat{\mu}(s) = \hat{C}(s) \tag{28}$$

Whose roots are:

$$\hat{C}(s) = \frac{\hat{\omega}(s)}{3} \tag{29}$$

$$\hat{\mu}(s) = \frac{\omega(s)}{6} \tag{30}$$

So:

$$\nu \rightarrow \frac{6\nu_0 + E_0 s\hat{\omega}(s)}{6 + 2E_0 s\hat{\omega}(s)} \tag{31}$$

The creep function in shear can be written as:

$$J_G = \frac{1}{G_0} + \omega(t) \tag{32}$$

Where:

$$\omega(t) = \frac{1}{G}\left(1 - e^{\frac{-Gt}{\eta}}\right) \tag{33}$$

$$\omega(t) = \frac{1}{G}\left(1 - e^{\frac{-Gt}{\eta}}\right) \tag{34}$$

$\eta =$ coefficient of viscosity.

Therefore we can write:

$$r = qBI\,\frac{(1+\nu)2}{2E}(1-\nu) = qBI\,\frac{(1-\nu)}{2G} \tag{35}$$

$$\hat{r}(s) = \frac{qBI\left(\frac{1}{G_0} + \frac{1}{G+\eta s}\right)\left(1 - \frac{6\nu_0 + \frac{2G_0(1+\nu)}{G+\eta s}}{6 + \frac{4G_0(1+\nu_0)}{G+\eta s}}\right)}{2s} \tag{36}$$

Performing the reverse transform and developing it, we obtain:

$$r(t) = qBI\left\{A + \frac{\left(\frac{C}{3}\right)(D) + (E)e^{\left(\frac{C}{3\eta}\right)^t}}{(C)(F) + 20G^3 + 8G^3\nu_0}\right\} \tag{37}$$

Where:

$$A = \frac{7 - 5\nu_0 + (1+\nu_0)}{G(10 + 4\nu_0)} - \frac{1}{4G}e^{\left(\frac{-Gt}{\eta}\right)};$$

$$C = (-5G - 2G\nu_0);$$

$$D = 20G + 2(C) - 16G\nu_0 - 2\nu_0(C);$$

$$E = 14G^2 - 10G^2\nu_0 + 2G^2(1+\nu_0);$$

$$F = \frac{64G^2}{3} + \frac{16G^2\nu_0}{3} + 4G(C).$$

A footing width of 1.2 m and average applied stress 174.2 kN / m² were adopted to illustrate it.

It is important to mention that the parameter for coefficient of viscosity (η) of the model was accepted at value 1000 kPa/day [8]. The average value 10615 kN/m² was used for the transverse elastic modulus (G_0). This value was adopted based on Poisson's ratio (V_0) equals 0.4 and average elastic modulus value (E) for soil equals 27600 kN/m².

Fig. 3 shows settlement evolution estimated by TLV over time.

Settlement growth can be observed over time based on Fig. 3. After about 300 days, this settlement shows a tendency for stabilization. Concerning the data, the estimated settlement from the TVL reached approximately 6.4 mm in 1043 days.

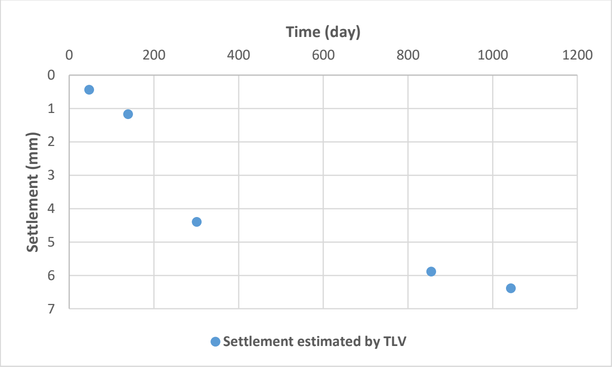


Fig. 3 Settlement evolution estimated by TLV over time.

4. Conclusions

The following conclusions are drawn:

(i) The proposed methodology allows the estimation of settlement over time for a spread footing underneath the center of compression load area;

(ii) Settlement growth is observed over time based on the application example, the growth of settlement is observed over time. The estimated values are always very small, reaching 6.4 mm in 1043 days. This value is directly related to the footing width, average applied stress, coefficient of viscosity, transverse elastic modulus, Poisson's coefficient and soil elastic modulus;

(iii) After about 300 days settlement tends to stabilize.

References

[1] D. R. Bland, *Theory of Linear Viscoelasticity*, Pergamon Press, 1960.

[2] A. D. Dozdrov, Finite Elasticity and Viscoelasticity: A Course in the Nonlinear Mechanics of Solids. Massachusettes, Word Scientific Publishing Co. Pte. Ltd. (1996).

[3] R. M. Cristensen, *Theory of Viscoelasticity* (2nd ed.), N. Y., Dover Publication Inc., 2003.

[4] W. Flügge, *Viscoelasticity*, Blaisdell Publishing Company, 1967.

[5] F. L. L. B. Carneiro, *Considerações sobre a Influência da Retração e Fluência do Concreto no Cálculo das Estruturas*, Colóquio sobre Retração e Deformação Lenta do Concreto, IBRACON, São Paulo, Brasil, 1978, pp. 1-23.

[6] F. C. M. Santa Maria, *A Viscoelasticidade Linear Aplicada a Problemas de Adensamento e Interação Solo-Estrutura*, Exame de Qualificação ao Doutorado, COPPE/UFRJ, Rio de Janeiro, 1997.

[7] W. N. Findley, J. S. Lai, and K. Onaran, *Creep and Relaxation of Nonlinear Viscoelastic Materials*, North-Holland Publishing Company, 1976.

[8] S. S. Vyalov, *Rheological Fundamentals of Soil Mechanics* (1st ed.), Elsevier Science, 1986, p. 564.